Coarse geometry: property A and coarse embeddability into a Hilbert Space

In this talk, we will examine two fundamental properties of coarse spaces: property A and coarse embeddability into a Hilbert space. These properties play a crucial role in the coarse Baum-Connes conjecture and the rigidity problem for Roe algebras. We will demonstrate that they are preserved under coarse equivalences and present illustrative examples. Their significance for the rigidity problem will be addressed in the third talk.

2 Coarse geometry: property A and coarse embeddability into Hilbert space

In this talk, we will introduce two invariants of coarse spaces, namely property A and coarse embeddability into a Hilbert space. We start with property A. As noted by Bruno de Mendonça Braga in his book "Large Scale Geometry in Functional Analysis" Property A, akin to amenability, has $2^{2^{\aleph_0}}$ equivalent characterisations. This talk will not encapsulate them all.

Definition 2.1 (Higson-Roe)**.** A coarse space $(X, \mathcal{E})$ has property A if for all $\varepsilon > 0$ and all $E \in \mathcal{E}$ there exists a function $\zeta \colon X \to \partial B_{l^1(X)}$ such that

1. $\|\zeta_x - \zeta_y\| \leq \varepsilon$ for all pairs $(x, y) \in E$;
2. The set $\{(x, y) \mid \zeta_x(y) \neq 0\}$ is controlled.

One can view ζ as a function from $X \times X$ to the complex numbers. In this case, the second condition states that ζ has controlled support, while the first condition is a certain condition on the variation of ζ in the first coordinate. For convenience, let's also provide the Higson-Roe definition for metric spaces.

Definition 2.2 (metric version of Higson-Roe)**.** A metric space (X, d) has property A if for all $\varepsilon > 0$ and all $R \geq 0$ there exists a function $\zeta \colon X \to \partial B_{l^1(X)}$ such that

1. $\|\zeta_x - \zeta_y\| \leq \varepsilon$ for all $x, y \in X$ satisfying $d(x, y) \leq R$;
2. The quantity $\sup_{x,y}\{d(x, y) \mid \zeta_x(y) \neq 0\}$ is finite.

We begin our study of property A with examples. The first of those should illustrate how to build the function ζ, while the latter ones aim at providing a huge variety of spaces with property A. As all bounded coarse spaces are coarsely equivalent to a point, we expect all of them to satisfy property A, as a point trivially does satisfy it.

2.3 (Example)**.** Bounded metric spaces satisfy property A trivially. Indeed, pick a unit norm vector $v \in l^1(X)$, and define $\zeta_x = v$. The first condition is satisfied for all $\varepsilon > 0$ and $E \in \mathcal{E}$, while the second condition is satisfied as every subset of $X \times X$ is controlled.

Now, we at least see that we are not talking about the empty set of spaces. Probably, the second easiest coarse space to try is $\mathbb{N}$ endowed with the Euclidean metric.

2.4 (Example)**.** Let $(\mathbb{N}, d)$ be a metric space whose underlying set is the set of natural numbers equipped with the Euclidean metric. For given $R \geq 0$ and $\varepsilon > 0$ we want to construct a sequence of norm-1 vectors $\zeta_n \in l^1(\mathbb{N})$ satisfying the following conditions:

1. $\|\zeta_n - \zeta_m\| \leq \varepsilon$ for all $n, m \in \mathbb{N}$ satisfying $|n - m| \leq R$;
2. The quantity $\sup_{n,m}\{|n - m| \mid \zeta_n(m) \neq 0\}$ is finite.

Pick $S \in \mathbb{N}$, and consider the following vectors

$$\zeta_n = \frac{1}{S}\mathbb{1}_{[n,n+S]}$$

The second condition of the definition is satisfied, as $\zeta_n(m) \neq 0$ means that $m \in [n, n+S]$, hence $|n - m| \leq S$. For the first condition, note that for $n, m \in \mathbb{N}$ such that $|n - m| \leq R$ one has

$$\|\zeta_n - \zeta_m\| = \frac{1}{S}\|\mathbb{1}_{[n,n+S]} - \mathbb{1}_{[m,m+S]}\| = \frac{1}{S}|[n, n+S] \triangle [m, m+S]| \leq \frac{2R}{S}$$

Pick S such that $2R/S < \varepsilon$, and we are done. Hence $(\mathbb{N}, d)$ has property A.

The previous example might have rung a bell with amenability of $\mathbb{Z}$. One may similarly prove that all amenable groups have property A.

2.5 (Example: Amenable groups)**.** Let G be a finitely generated group equipped with a shortest path metric of its Cayley graph. Suppose that G is amenable and let $\{F_i\}_{i\in\mathbb{N}}$ be a Følner sequence for G, i.e. a sequence $\{F_i\}_{i\in\mathbb{N}}$ of finite subsets of G such that:

1. For every $g \in G$ there exists $i \in \mathbb{N}$ such that $g \in F_j$ for all $j > i$;

2. For every $\varepsilon > 0$ and $g \in G$ there exists $i \in \mathbb{N}$ such that $|gF_j \triangle F_j| \leq \varepsilon |F_j|$ for all $j > i$.

Suppose given $\varepsilon > 0$ and $R \geq 0$. Pick $i \in \mathbb{N}$ big enough so that $|gF_j \triangle F_j| \leq \varepsilon |F_j|$ for all g satisfying $\gamma(g) \leq R$, i.e. for all g that can be written as a product of $\lceil R \rceil$ generators. Define a function ζ_g for $g \in G$ as follows.

$$\zeta_g = \frac{1}{|F_i|}\mathbb{1}_{gF_i}$$

The second condition is satisfied, as $\zeta_g(h) \neq 0$ means that $h \in gF_i$, so $d(g,h) \leq \sup_{x\in F_i} \gamma(x) < \infty$. For the first condition, note that

$$\|\zeta_g - \zeta_h\| = \frac{|gF_i \triangle hF_i|}{|F_i|} = \frac{|h^{-1}gF_i \triangle F_i|}{|F_i|} \leq \varepsilon,$$

for $g, h \in G$ such that $d(g,h) \leq R$. We are done.

There is a notion of amenability for metric spaces, though it does not imply property A in general. The class of groups with property A is not exhausted by amenable groups. For example, free groups have property A.

2.6 (Example: Free groups)**.** Let $\mathbb{F}_n$ be a free group on n generators. The Cayley graph of $\mathbb{F}_n$ is a $2n$-valent tree. Fix $R \geq 0$ and $\varepsilon > 0$. Let γ be an infinite ray in the Cayley graph of $\mathbb{F}_n$ (i.e. an image of an isometry $\tilde{\gamma}\colon \mathbb{N} \to \mathbb{F}_n$, where $\mathbb{N}$ is endowed with the Euclidean metric). For each $g \in \mathbb{F}_n$, let γ_g be the unique ray starting at g that follows the same path as γ. This ray is unique, as any tree is a uniquely geodesic metric space. The ray γ_g can be constructed as follows.

1. Suppose g belongs to the ray γ. Let $n \in \mathbb{N}$ be the preimage of g by $\tilde{\gamma}$. Define γ_g as the image of the isometry $\tilde{\gamma}\colon \mathbb{N} \setminus \{1, \dots, n\} \to \mathbb{F}_n$;

2. Suppose g does not belong to the ray γ. Let $h_g \in \gamma$ be the closest point to g, then define γ_g to be the concatenation of the unique geodesic from g to h_g and the ray γ_h defined previously.

For $g \in \mathbb{F}_n$ consider a set $A_g \subset \mathbb{F}_n$ that consists of $h \in \mathbb{F}_n$ such that $h \in \gamma_g$ and $d(g,h) \leq 3R/\varepsilon + 1$. Define a function

$$\zeta_g\colon \mathbb{F}_n \to \partial B_1(l^1(\mathbb{F}_n)), \quad \zeta_g(x) = \frac{1}{|A_g|}\mathbb{1}_{A_g}(x).$$

It remains to check that ζ_g satisfies the Higson-Roe definition. Suppose that $d(g,h) \leq R$. Note that

1. For all $g \in \mathbb{F}_n$ one has $3R/\varepsilon \leq |A_g| \leq 3R/\varepsilon + 1$;

2. For all $g, h \in \mathbb{F}_n$ such that $d(g,h) \leq R$, there are at most $2R$ elements in $A_g \triangle A_h$. Indeed, the elements of the symmetric difference are located either on the shortest path between g and h (and there are at most R of them) or at the tail of the ray. But since $|A_g| = |A_h|$ there are at most R elements of $A_g \triangle A_h$ at the tail. The conclusion follows;

Henceforth, we may establish the estimate for the norm of the difference of ζ_g and ζ_h:

$$\|\zeta_g - \zeta_h\|_1 = \sum_{x\in\mathbb{F}_n} \left|\frac{1}{|A_g|}\mathbb{1}_{A_g}(x) - \frac{1}{|A_h|}\mathbb{1}_{A_h}(x)\right| \leq \frac{|A_g \triangle A_h|}{|A_g|} \leq \frac{2R}{3R/\varepsilon} = \frac{2\varepsilon}{3} \leq \varepsilon$$

Therefore, the first assertion is proven. For the second assertion, by the first statement above $\operatorname{diam}(A_g) \leq 3R/\varepsilon + 1$, hence if for some $h \in \mathbb{F}_n$ the inequality $d(g,h) > 3R/\varepsilon + 1$ holds, then $\zeta_g(h) = 0$. Hence $\mathbb{F}_n$ has property A.

In the previous example, the only property of $\mathbb{F}_n$ we have used is that its Cayley graph is a tree. In particular, all trees have property A. Examples of space without property A are hard to build.

2.7 (Example: Space without Property A)**.** Let Γ be a nontrivial finite group. Let d be a metric on Γ induced by a Cayley graph of Γ and for every $n \in \mathbb{N}$ equip Γ^n with a product metric d_n (sum of metrics applied coordinatewise). Define a metric space (X, d) as a coarse disjoint union of $\{\Gamma^n\}_{n\in\mathbb{N}}$, i.e.:

$$X = \bigsqcup_{n\in\mathbb{N}} \Gamma^n, \quad d(g,h) = \begin{cases} d_n(g,h), & \text{if } g,h \in \Gamma^n; \\ |k^2 - l^2|, & \text{if } g \in \Gamma^k \text{ and } h \in \Gamma^l, \text{ for } k \neq l. \end{cases}$$

Then (X, d) does not have property A.

Expander graphs constitute another example of such spaces. As we shall see shortly, a subspace of a property A space has property A. Hence, every coarse space where an expander graph can be embedded does not possess property A. For example, the Gromov Monster group does not have property A. Recall that the Thompson group F is a finitely generated group given by

$$F = \langle x_0, x_1, x_2, \dots \mid x_k^{-1} x_n x_k = x_{n+1} \text{ for } k < n \rangle = \langle a, b \mid [ab^{-1}, a^{-1}ba] = [ab^{-2}, a^{-1}ba^2] = 1 \rangle .$$

The question of whether the Thompson group F has property A is open. We haven't yet proven that property A is a coarse invariant. I.e., suppose given two coarsely equivalent coarse spaces, one of which possesses property A, is it true that the second one has property A as well?

Theorem 2.8. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces and $f\colon X \to Y$ be a coarse embedding. If $(X, \mathcal{E})$ has property A, then $(Y, \mathcal{F})$ has property A as well.

Sketch of proof. For simplicity, assume that (X, d) and (Y, ∂) are metric spaces. Suppose that $f\colon X \to Y$ is injective. Suppose given $\varepsilon > 0$ and $R \geq 0$. Since f is coarse there exists $R' \geq 0$ such that whenever $d(x,y) \leq R$ one has $\partial(f(x), f(y)) \leq R'$. Since (Y, ∂) has property A there exists a function

$$\zeta^Y : Y \to \partial B_{l^1(Y)}$$

that satisfies the two properties from the definition for $\varepsilon > 0$ and $R' \geq 0$. Note that ζ^Y can be picked to be a positive function. By composing f and ζ^Y, we get a map from X to $\partial B_{l^1(Y)}$. Let $S \geq 0$ be a quantity such that whenever $\zeta^Y_x(y) \neq 0$ one has $d(x,y) \leq S$ and consider the S-neighbourhood $f(X)_S$ of $f(X)$. Since f is cobounded there exists a coarse retraction $p\colon f(X)_S \to f(X)$, i.e. a map such that $p \circ i(t) = t$ for every $t \in \mathrm{im}(f)$ and $\partial(y, p(y)) \leq S$, for all $y \in Y$. Define

$$\zeta^X\colon X \to B_{l^1(X)}, \qquad \zeta^X_x(z) = \sum_{y \in p^{-1}(\{f(z)\})} \zeta^Y_{f(x)}(y).$$

Note that ζ^X_x has norm one for every $x \in X$. Indeed

$$\|\zeta^X_x\| = \sum_{z\in X} |\zeta^X_x(z)| = \sum_{z\in X} \sum_{y\in p^{-1}(f(z))} |\zeta^Y_{f(x)}(y)| = \sum_{y\in Y} |\zeta^Y_{f(x)}(y)| = \|\zeta^Y_{f(x)}\| = 1.$$

It remains to check that ζ^X satisfies the required conditions for $\varepsilon > 0$ and $R \geq 0$, this is left as an excersise. Hence every injective coarse embedding preserves property A. For general coarse embedding $f\colon X \to Y$ recall that f is expansive, hence there exists $K \geq 0$ such that

$$d(x,y) \geq K \quad \text{implies} \quad f(x) \neq f(y) \qquad (\text{we have considered } (f \times f)^{-1}(\Delta_Y)).$$

By Zorn lemma there exists a maximal K-separated subset $N \subset X$, i.e. a subset of X every two diferent points of which are at distance K from one to another. Now $f\colon N \to Y$ is an injective coarse embedding, hence N has property A by previously prooven statement. Let $\zeta^N\colon N \to \partial B_{l^1(N)}$ be a function provided by property A for $R + 2K \geq 0$ and $\varepsilon > 0$. Consider a coarse retraction $p\colon X \to N$, i.e, a map that sends a point $x \in N$ to itself and a point $x \in X \setminus N$ to a point $p(x) \in N$ such that $d(x, p(x)) \leq K$. Define

$$\zeta^X\colon X \to \partial B_{l^1(X)}, \quad x \mapsto \zeta^N_{p(x)}.$$

It remains to check that ζ^X satisfies the two conditions from the definition for $\varepsilon > 0$ and $R \geq 0$. □

Applying the above theorem two times we get a proof of the fact that property A is a coarse invariant. Another consequence of the above theorem is that a subspace of a coarse space with property A has property A. Property A was shown instrumental in the study of coarse Baum-Connes conjecture and the rigidity problem for Roe algebras. We will see the latter application in the next talk. We switch to the study of embeddability of coarse spaces into a Hilbert space. First, let's define it properly.

Definition 2.9. Let $(X, \mathcal{E})$ be a coarse space. We say that $(X, \mathcal{E})$ is embeddable into a Hilbert space if for some Hilbert space H there exists a coarse embedding $f\colon X \to H$, where the coarse structure on H is induced by the metric $d(v, w) = \|v - w\|$, $v, w \in H$.

It is automatic that coarsely equivalent spaces satisfy this property simultaneously, as any coarse equivalence is a coarse embedding and composition of coarse embeddings is a coarse embedding. Moreover, a subspace of a coarse space which is coarsely embeddable into a Hilbert space is also coarsely embeddable into a Hilbert space.
Note that having a coarse embedding into a metric space already implies that the coarse space is metrisable. Hence, all coarse spaces which are coarsely embeddable into a Hilbert space are metric. We will show that any property A metric space is coarsely embeddable into a Hilbert space. Before we need a preliminary result on the metric structure of unit balls of ℓ^p-spaces.

Lemma 2.10. Let μ be a measure on a measurable space X and $p, q \in [1, \infty)$, then the unit spheres of $L^p(X, \mu)$ and $L^q(X, \mu)$ are coarsely equivalent.

Sketch of proof. Without loss of generality, let $q \leq p$. Define a Mazur map

$$M_{p,q}\colon \partial B_{L^p(X,\mu)} \to \partial B_{L^q(X,\mu)}, \quad f \mapsto \operatorname{sign}(f)|f|^{p/q}.$$

Clearly for any $f \in \partial B_{L^p(X,\mu)}$ its image belongs to the unit sphere of $L^q(X, \mu)$, since

$$\|M_{p,q}(f)\|_q = (\int_X |f|^p d\mu)^{1/q} = \|f\|_p^{p/q} = 1.$$

One can show that for $f, g \in \partial B_{L^p(X,\mu)}$ the following estimate holds:

$$\frac{1}{2^{p/q}}\|f - g\|_p^{p/q} \leq \|M_{p,q}(f) - M_{p,q}(g)\|_q \leq \frac{p}{q}\|f - g\|_p.$$

This estimate precisely establishes that $M_{p,q}$ is a coarse map which is expansive. It is a bijection, since $M_{q,p} \circ M_{p,q} = \mathrm{id}$. ☐

As the unit spheres of L^p-spaces are the same, we may reformulate the Higson-Roe definition in terms of the unit spheres of l^p-spaces.

Definition 2.11. (Higson-Roe) A metric space (X, d) has property A if for all $\varepsilon > 0$ and all $R \geq 0$ there exists a function $\zeta\colon X \to \partial B_{l^p(X)}$ such that

1. $\|\zeta_x - \zeta_y\|_p \leq \varepsilon$ for all $x, y \in X$ satisfying $d(x, y) \leq R$;
2. The quantity $\sup_{x,y}\{d(x, y) \mid \zeta_x(y) \neq 0\}$ is finite.

Indeed, suppose (X, d) satisfies property A for $p = 1$, let ζ^1 be the function provided by the definition. Define $\zeta^p = M_{1,p} \circ \zeta^1$, then the second condition is automatic, while the first condition follows from the nonproven estimate. Vise-versa, the same argument applies to $\zeta^1 = M_{p,1} \circ \zeta^p$.

Theorem 2.12. Let (X, d) be a coarse space with property A, then (X, d) is coarsely embeddable into a Hilbert space.

Proof. Since (X, d) has property A, for every $n \in \mathbb{N}$ there is a function $\zeta^n\colon X \to \partial B_{\ell^2(X)}$ such that:

1. $\|\zeta^n_x - \zeta^n_y\| \leq 1/2^n$ for all $x, y \in X$ satisfying $d(x, y) \leq n$;
2. The quantity $s_n = \sup\{d(x, y) \mid \zeta^n_x(y) \neq 0\}$ is finite.

Fix $y \in X$ and define a function

$$f\colon X \to \ell^2(\mathbb{N}) \otimes \ell^2(X), \quad f(x) = \sum_{n=1}^{\infty} \delta_n \otimes (\zeta^n_x - \zeta^n_y).$$

It follows from the two conditions above that f is well-defined. Hence, it is enough to check that f is a coarse embedding. For $x, z \in X$ satisfying $d(x, z) \leq R$ one has

$$\|f(x) - f(z)\| = \sum_{n=1}^{\infty} \|\zeta_x^n - \zeta_z^n\|^2 = \sum_{n \leq R}^{\infty} \|\zeta_x^n - \zeta_z^n\|^2 + \sum_{n > R}^{\infty} \|\zeta_x^n - \zeta_z^n\|^2 \leq 4R + 1.$$

It follows that f is coarse. Note that being expansive is equivalent to saying that for every $S \geq 0$, there exists $R \geq 0$ such that whenever $d(x, y) \geq R$, one has $\|f(x) - f(y)\| \geq S$. Fix $S \geq 0$ and note that by the second condition above, if $d(x, y) \geq 2s_n$, then the supports of ζ_x^n and ζ_y^n are disjoint. Therefore, for all $x, y \in X$ with $d(x, y) > 2s_n$ one has

$$\|\zeta_x^n - \zeta_y^n\|^2 = 2.$$

Fix $k \in \mathbb{N}$ such that $2k \geq s$. Define $R = \max\{2s_n \mid n \leq k\} + 1$, then for all $x, y \in X$ such that $d(x, y) \geq R$ one has

$$\|f(x) - f(y)\| = \sum_{n=1}^{\infty} \|\zeta_x^n - \zeta_y^n\|^2 \geq \sum_{n=1}^{k} \|\zeta_x^n - \zeta_y^n\|^2 \geq 2k \geq s.$$

It follows that f is expansive. Hence, it is a coarse embedding. □

There are coarse spaces which does not have property A, but they are embeddable into a Hilbert space. There are also examples of coarse spaces which are not coarsely embeddable into a Hilbert space. As it was announced, these properties are crucial for the coarse Baum-Connes conjecture. If (X, d) is a uniformly locally finite metric space that embeds coarsely into a Hilbert space, then the coarse Baum-Connes conjecture holds for (X, d). The applications of these properties to the rigidity problem of Roe algebras will be shown in the next talk.